Investment Analysis (part 1)—Exam

Paulo Maio

Spring 2013

1 Instructions

1. Maximum score is 100 points.
2. The duration is 4 hours.
3. There are 3 groups of questions with the number of points indicated in parenthesis.
4. Each group should be answered in a separate paper sheet.
5. Each paper should have your name in the top.
6. The answers should be written in a clear way!

2 Questions

2.1 CAPM (40 points)

1. Given the SDF, M_{t+1}, and the gross return on an arbitrary asset, $R_{i,t+1}$, derive the expected return-covariance representation.
2. Now consider the SDF for the CAPM,

$$M_{t+1} = a + bR_{m,t+1},$$

where $R_{m,t+1}$ is the gross market return and assume that $E(R_{m,t+1}) = 0$. The SDF representation for $R_{i,t+1}$ is given by

$$1 = E(M_{t+1}R_{i,t+1}).$$

By using the expression obtained in (1), find the equivalent expected return-beta representation,

$$E(R_{i,t+1}) = \gamma + \lambda\beta_i,$$

where β_i is the market beta for asset i's return, and γ and λ are constants to be determined.

3. Assume now the expected (excess) return-beta representation for the CAPM,

$$E(R^e_{i,t+1}) = \lambda\beta_i$$

where $R^e_{i,t+1}$ is the excess return for asset i, β_i denotes the (excess return) market beta for asset i, and λ denotes the market price of risk. Derive the SDF representation,

$$\begin{aligned} 0 &= E(M_{t+1}R^e_{i,t+1}), \\ M_{t+1} &= E(M_{t+1}) + bR_{m,t+1} \end{aligned}$$

where b is a constant to be determined, and again assume $E(R_{m,t+1}) = 0$. Can we identify $E(M_{t+1})$ in this case?

4. Derive the relation between the expected market return and the respective variance. Interpret your results.

5. Now assume that $E(R_{m,t+1}) \neq 0$, and also assume that there is a risk-free asset with gross return, $R_{f,t+1}$. Find an expression for both a and b as a function of $E(R_{m,t+1})$, $R_{f,t+1}$, and the variance of the market return, σ^2_m.

2.2 Answer

1. Starting with the SDF pricing equation for $R_{i,t+1}$, we obtain,

$$\begin{aligned} 1 &= E(M_{t+1}R_{i,t+1}) \Leftrightarrow \\ 1 &= Cov(M_{t+1}, R_{i,t+1}) + E(R_{i,t+1})E(M_{t+1}) \Leftrightarrow \\ E(R_{i,t+1}) &= \frac{1}{E(M_{t+1})} - \frac{Cov(R_{i,t+1}, M_{t+1})}{E(M_{t+1})}, \end{aligned}$$

which represents the expected return-covariance representation.

2. The mean SDF is given by

$$E(M_{t+1}) = a + b\,E(R_{m,t+1}) = a,$$

and by substituting $M_{t+1} = a + bR_{m,t+1}$ and $E(M_{t+1}) = a$ in the expected return-covariance equation derived in (1), we obtain:

$$\begin{aligned} E(R_{i,t+1}) &= \frac{1}{a} - b\frac{Cov(R_{i,t+1}, R_{m,t+1})}{a} \\ &= \frac{1}{a} - b\frac{Var(R_{m,t+1})}{a}\frac{Cov(R_{i,t+1}, R_{m,t+1})}{Var(R_{m,t+1})} \\ &= \gamma + \lambda\beta_i, \end{aligned}$$

where

$$\begin{cases} \gamma \equiv \frac{1}{a}, \\ \lambda \equiv -b\frac{Var(R_{m,t+1})}{a}, \\ \beta_i = \frac{Cov(R_{i,t+1}, R_{m,t+1})}{Var(R_{m,t+1})}. \end{cases} \tag{1}$$

3. By assuming that

$$\lambda = -b\frac{Var(R_{m,t+1})}{E(M_{t+1})},$$

and substituting in the beta equation, we obtain,

$$\begin{aligned} E(R^e_{i,t+1}) &= \lambda\beta_i \Leftrightarrow \\ E(R^e_{i,t+1}) &= -b\frac{Var(R_{m,t+1})}{E(M_{t+1})}\frac{Cov(R^e_{i,t+1}, R_{m,t+1})}{Var(R_{m,t+1})} \Leftrightarrow \\ E(R^e_{i,t+1}) &= -b\frac{Cov(R^e_{i,t+1}, R_{m,t+1})}{E(M_{t+1})} \Leftrightarrow \\ E(R^e_{i,t+1}) &= -\frac{Cov(R^e_{i,t+1}, bR_{m,t+1})}{E(M_{t+1})} \Leftrightarrow \\ 0 &= E(R^e_{i,t+1})E(M_{t+1}) + Cov[R^e_{i,t+1}, E(M_{t+1}) + bR_{m,t+1}] \Leftrightarrow \\ 0 &= E(R^e_{i,t+1})E(M_{t+1}) + Cov(R^e_{i,t+1}, M_{t+1}) \Leftrightarrow \\ 0 &= E(M_{t+1}R^e_{i,t+1}). \end{aligned}$$

Assume $E(M_{t+1}) = 0$ and $M_{t+1} = E(M_{t+1}) + bR_{m,t+1}$

which is the SDF pricing equation for $R_{i,t+1}$, and $M_{t+1} = \mathrm{E}(M_{t+1}) + bR_{m,t+1}$. We have:

$$b = -\lambda \frac{\mathrm{E}(M_{t+1})}{\mathrm{Var}(R_{m,t+1})},$$

and thus, $\mathrm{E}(M_{t+1})$ is not identifiable in this case.

4. By applying the beta equation to the market return, and since the beta for the market portfolio is one, we have:

$$\mathrm{E}(R_{m,t+1}) = \gamma + \lambda = \frac{1}{a} - b\frac{\mathrm{Var}(R_{m,t+1})}{a}.$$

It follows that

$$\frac{\partial\, \mathrm{E}(R_{m,t+1})}{\partial\, \mathrm{Var}(R_{m,t+1})} = -\frac{b}{a} > 0,$$

under the assumption that $b < 0$. The intuition is that a higher market variance means a lower covariance between the SDF and the market return if $b < 0$, and that translates into a higher expected market return because the aggregate portfolio offers a higher return when marginally utility is relatively low, that is, in good times.

5. The pricing equation for $R_{f,t+1}$ is given by

$$\frac{1}{R_{f,t+1}} = a + b\mathrm{E}(R_{m,t+1}) \Leftrightarrow a = \frac{1}{R_{f,t+1}} - b\mathrm{E}(R_{m,t+1}),$$

and the SDF equation for the market return is as follows:

$$\begin{aligned} 1 &= \mathrm{E}[(a + bR_{m,t+1})R_{m,t+1}] = a\,\mathrm{E}(R_{m,t+1}) + b\,\mathrm{E}(R^2_{m,t+1}) \\ &= a\,\mathrm{E}(R_{m,t+1}) + b[\sigma^2_m + \mathrm{E}(R_{m,t+1})^2]. \end{aligned}$$

By substituting the expression for a derived above, we obtain

$$1 = \left[\frac{1}{R_{f,t+1}} - b\,\mathrm{E}(R_{m,t+1})\right]\mathrm{E}(R_{m,t+1}) + b[\sigma_m^2 + \mathrm{E}(R_{m,t+1})^2] \Leftrightarrow$$
$$1 = \frac{\mathrm{E}(R_{m,t+1})}{R_{f,t+1}} + b\sigma_m^2 \Leftrightarrow$$
$$b = -\frac{\mathrm{E}(R_{m,t+1}) - R_{f,t+1}}{R_{f,t+1}\sigma_m^2},$$

and by substituting back in the expression for a leads to

$$a = \frac{\sigma_m^2 + \mathrm{E}(R_{m,t+1})\,[\mathrm{E}(R_{m,t+1}) - R_{f,t+1}]}{R_{f,t+1}\sigma_m^2}.$$

2.3 Exponential utility (35 points)

Consider a one-period representative investor with exponential utility.

$$U(W) = -e^{-bW},$$

where W represents wealth. There are two assets in this economy—a risky asset with gross return R and a risk-free asset with gross return R_f. The gross return on the investor's portfolio is denoted by R_p. ω represents the portfolio weight on the risky asset. The initial level of wealth is W_0, and the wealth at the end of the period is given by W_1. Assume that the risky asset has expected return of μ and variance of return of σ^2. The investor maximizes the expected utility of wealth.

1. Derive the absolute risk aversion, relative risk aversion associated with W_0 (b_r); the utility function in terms of R_p; and the equation for R_p in terms of the individual returns.

2. Given the expressions derived in (1) define the investor's problem and derive the optimal portfolio weight, ω^*. Interpret your results.

3. Derive the CAPM beta equation in this economy. Interpret your results.

4. Consider now that there is a second risky asset with gross return, $\hat{R}$. The expected return and variance of this asset are denoted by $\hat{\mu}$ and $\hat{\sigma}^2$, respectively. Assume that the two risky assets are uncorrelated. Denote the portfolio weight on the second asset by $\hat{\omega}$. Determine the mean-variance optimal weights on the two risky assets for a target expected return of μ_p. Interpret your results.

2.4 Answer

1. The absolute risk aversion is equal to

$$ARA(W_0) = -\frac{U''(W_0)}{U'(W_0)} = -\frac{-b^2e^{-bW_0}}{be^{-bW_0}} = b,$$

the relative risk aversion associated with W_0 is given by

$$b_r = RRA(W_0) = W_0 ARA(W_0) = W_0 b.$$

the utility function in terms of R_p is

$$U(W_1) = -e^{-bW_1} = -e^{-b_r \frac{W_1}{W_0}} = -e^{-b_r R_p},$$

and the portfolio return is as follows:

$$R_p = R_f + \omega(R - R_f).$$

2. The utility function at time 1 can be rewritten as

$$U(W_1) = -e^{-b_r R_p} = -e^{-b_r[R_f + \omega(R - R_f)]},$$

and the expected utility is given by

$$\begin{aligned} \mathbb{E}[U(W_1)] &= \mathbb{E}\left\{-e^{-b_r[R_f + \omega(R - R_f)]}\right\} \\ &= -e^{-b_r[R_f + \omega(\mu - R_f)] + \frac{1}{2}b_r^2\omega^2\sigma^2}. \end{aligned}$$

The investor's problem is,

$$\max_{\omega} -e^{-\delta_r[R_f+\omega(\mu-R_f)]+\frac{1}{2}\delta_r^2\omega^2\sigma^2} \Leftrightarrow$$
$$\max_{\omega} \delta_r[R_f+\omega(\mu-R_f)] - \frac{1}{2}\delta_r^2\omega^2\sigma^2,$$

and the first order condition leads to the optimal portfolio weight:

$$\delta_r(\mu - R_f) - \delta_r^2\omega\sigma^2 = 0$$
$$\omega^* = \frac{\mu - R_f}{\delta_r\sigma^2}.$$

The optimal weight increases with the risk premium, $\mu - R_f$, and decreases with both the relative risk aversion coefficient (δ_r) and the return volatility (σ^2).

3. By rearranging the equation for the optimal portfolio weight, we obtain,

$$\begin{aligned}
\mu - R_f &= \omega\delta_r\sigma^2 \\
&= \delta_r \operatorname{Cov}(R, R)\omega \\
&= \delta_r \operatorname{Cov}(R, \omega R) \\
&= \delta_r \operatorname{Cov}[R, R_f + \omega(R - R_f)] \\
&= \delta_r \operatorname{Cov}(R, R_p),
\end{aligned}$$

which corresponds to the CAPM covariance equation with R_p as the reference (market) portfolio return. Since δ_r is always positive, we have that an asset that covaries positively with the portfolio return, $\operatorname{Cov}(R, R_p) > 0$, has a positive risk premium, $\mu - R_f > 0$.

The corresponding beta equation is given by

$$\mu - R_f = \delta_r \operatorname{Var}(R_p)\frac{\operatorname{Cov}(R, R_p)}{\operatorname{Var}(R_p)} = \lambda\beta,$$

where $\lambda \equiv \delta_r \operatorname{Var}(R_p)$ denotes the market risk price.

and by substituting in the f.o.c. for the first asset, we obtain the optimal weight:

$$\begin{aligned}\omega &= \frac{\mu - R_f}{\sigma^2}\lambda \\ &= (\mu - R_f)\frac{\mu_p - R_f}{(\mu - R_f)^2 + \frac{\sigma^2}{\widehat{\sigma}^2}(\widehat{\mu} - R_f)^2}.\end{aligned}$$

and similarly the optimal weight for the second asset is given by

$$\widehat{\omega} = (\widehat{\mu} - R_f)\frac{\mu_p - R_f}{(\mu - R_f)^2\frac{\sigma^2}{\widehat{\sigma}^2} + (\widehat{\mu} - R_f)^2}.$$

We can see that the optimal weight on each risky asset increases with the own risk premium and the target mean portfolio return, while it decreases with the risk premium on the other asset. Similarly, the optimal weight decreases with the own variance, but increases with the variance of the other asset.

2.5 Consumption CAPM (25 points)

Let $R_{i,t+1}$ be the gross-return on an arbitrary asset i, $R_{f,t+1}$ the gross-risk free rate, and M_{t+1} the SDF in this economy. Assume that there is no conditioning information. Assume that the log return, log risk-free rate, and log SDF are given by $r_{i,t+1}$, $r_{f,t+1}$, and m_{t+1}, respectively.

1. By assuming that the returns and the SDF are jointly log-normal, derive the expected log excess return-covariance representation of the asset pricing model associated with the SDF, M_{t+1}.

2. Now assume that the SDF is given by

$$M_{t+1} = \delta\frac{U'(C_{t+1})}{U'(C_t)},$$

where $U'(\cdot)$ is the marginal utility of consumption; δ is a time-preference parameter; and the investor has log utility. By using the pricing equation derived in (1), derive the expected log excess return-covariance representation for the consumption-

CAPM in this case. What is the price of risk for the log consumption beta? Interpret your results.

2.6 Answer

1. The SDF equation for the risky asset can be written as

$$1 = \mathrm{E}\left(e^{m_{t+1}+r_{i,t+1}}\right) \Leftrightarrow$$
$$0 = \ln\left[\mathrm{E}\left(e^{m_{t+1}+r_{i,t+1}}\right)\right].$$

If $M_{t+1}, R_{i,t+1}$ are jointly log normal, it follows

$$0 = \mathrm{E}\left(m_{t+1} + r^i_{t+1}\right) + 0.5\,\mathrm{Var}\left(m_{t+1} + r^i_{t+1}\right) \Leftrightarrow$$
$$\mathrm{E}\left(r_{i,t+1}\right) + 0.5\,\mathrm{Var}\left(r_{i,t+1}\right) = -\,\mathrm{E}\left(m_{t+1}\right) - \frac{1}{2}\,\mathrm{Var}\left(m_{t+1}\right) - \mathrm{Cov}\left(m_{t+1}, r_{i,t+1}\right),$$

and the corresponding pricing equation for the log risk-free rate is given by

$$r_{f,t+1} = -\,\mathrm{E}\left(m_{t+1}\right) - \frac{1}{2}\,\mathrm{Var}\left(m_{t+1}\right).$$

By combining the previous two equations, we obtain the expected log (excess) return-covariance representation,

$$\mathrm{E}(r_{i,t+1}) - r_{f,t+1} + \frac{1}{2}\,\mathrm{Var}(r_{i,t+1}) = -\,\mathrm{Cov}(m_{t+1}, r_{i,t+1}).$$

2. With $U(C_t) = \ln(C_t)$, the SDF is given by

$$M_{t+1} = \delta\frac{U'(C_{t+1})}{U'(C_t)} = \delta\frac{C_t}{C_{t+1}},$$

and the corresponding log SDF is

$$m_{t+1} = \ln(\delta) - \ln\left(\frac{C_{t+1}}{C_t}\right) = \ln(\delta) - \Delta c_{t+1}.$$

By substituting in the general covariance equation derived in (1), we obtain:

$$\begin{aligned} E(r_{i,t+1}) - r_{f,t+1} + \frac{1}{2}\mathrm{Var}(r_{i,t+1}) &= \mathrm{Cov}(r_{i,t+1}, \Delta c_{t+1}) \\ &= \mathrm{Var}(\Delta c_{t+1})\beta_{i,c} \end{aligned}$$

where $\beta_{i,c} \equiv \mathrm{Cov}(r_{i,t+1}, \Delta c_{t+1}) / \mathrm{Var}(\Delta c_{t+1})$ is the consumption beta. In this version of the consumption-CAPM, the covariance price of risk is one, and the beta price of risk is the variance of log consumption growth. This represents a special case of the consumption-CAPM with power utility.

4. The problem is

$$\min_{\omega,\widehat{\omega}} \omega^2\sigma^2 + \widehat{\omega}^2\widehat{\sigma}^2$$
$$\text{s.t. } \mu_p = R_f + \omega(\mu - R_f) + \widehat{\omega}(\widehat{\mu} - R_f),$$

and the Lagrange function is given by

$$L = \omega^2\sigma^2 + \widehat{\omega}^2\widehat{\sigma}^2 + 2\lambda(\mu_p - R_f - \omega(\mu - R_f) - \widehat{\omega}(\widehat{\mu} - R_f)).$$

where λ is the Lagrange multiplier.

It follows that the f.o.c. for the first risky asset is

$$2\omega\sigma^2 - 2\lambda(\mu - R_f) = 0 \Leftrightarrow$$
$$\omega = \frac{\mu - R_f}{\sigma^2}\lambda.$$

and similarly for the second risky asset:

$$\widehat{\omega} = \frac{\widehat{\mu} - R_f}{\widehat{\sigma}^2}\lambda.$$

By substituting back in the constraint, we obtain an expression for the Lagrange multiplier,

$$\mu_p = R_f + \frac{\mu - R_f}{\sigma^2}\lambda(\mu - R_f) + \frac{\widehat{\mu} - R_f}{\widehat{\sigma}^2}\lambda(\widehat{\mu} - R_f) \Leftrightarrow$$
$$\mu_p - R_f = \lambda\left[\frac{(\mu - R_f)^2}{\sigma^2} + \frac{(\widehat{\mu} - R_f)^2}{\widehat{\sigma}^2}\right] \Leftrightarrow$$
$$\lambda = \frac{\mu_p - R_f}{\frac{(\mu - R_f)^2}{\sigma^2} + \frac{(\widehat{\mu} - R_f)^2}{\widehat{\sigma}^2}},$$